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Some considerations of finite-dimensional spin glasses

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Abstract

In this paper I will review the theoretical results that have been obtained for spin glasses, paying particular attention to finite-dimensional spin glasses. I will concentrate my attention on the formulation of the mean-field approach and on its numerical and experimental verifications. I will mainly consider equilibrium properties at zero magnetic field, where the situation is clear and it should not be controversial. I will present the various hypotheses on the basis of the theory and I will discuss their physical status.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this paper, after a very brief history of the subject, I will concentrate my attention on some of the main predictions of the replica approach [1–5], that clearly distinguish it from other approaches (e.g. from the droplet model [6] and from an intermediate possibility, the so-called TNT scenario [7]). I will compare the theoretical results with the data coming from simulations and briefly from experiments.

I will focus my attention on equilibrium results, briefly touching the very important region of off-equilibrium properties. A main role is the study of the correlations, both in the usual short-range model and in the one-dimensional model with long-range interactions. Finally, I will discuss the status of the properties of ultrametricity in three-dimensional short-range models. Some conclusions are presented at the end.

2. A brief history

2.1. Slow progress

Although the theoretical investigations of spin glasses have a long history, deep studies of spin glass theory started with the Edwards–Anderson paper [8] and the Sherrington–Kirkpatrick

(SK) papers [9]. The SK papers made clear that there was something very strange in this *soluble* model [10–12], whose *exact* solution was not correct, as far as it gave a negative entropy at low temperature. This difficulty was later solved [1], but the theoretical efforts have been quite extensive: they needed the introduction of many new concepts. The physical theory of complexity was born with this model.

The SK model is conceptually simple, it defines the mean-field theory of spin glasses. However the theory of spin glasses is difficult. Already in the mean-field framework it took a long time to clarify the theory.

- It took five years to solve a soluble model [1].
- It took five years to find the full physical meaning of the physical solution after it was found [1, 13].
- It took another ten years in order to find out the origin of some properties of the solution (e.g. stochastic stability [14–16]).
- It took 25 years to prove that the free-energy computation in mean field is correct [17, 18].
- It has not yet been mathematically proved that all computations of the observables done in mean-field theory are correct, although there are no doubts over their correctness.

In finite dimensions we know the correlation functions in the Curie–Weiss approximation [19]. Already this computation is a *tour de force*. The renormalization group (i.e. the study of the non-Gaussian fixed point) is still in its infancy. A full computation at one loop of the corrections to the Curie–Weiss approximation is missing (logs are presents in $D < 6$). Partial, but very instructive, computations have been done [19] from which we have gained a great insight. Ward identities have been identified [20], but we have not found a relevant nonlinear sigma model that should give the dominant contribution at low temperature. A simple argument has been presented that implies that the lower critical dimension is $D = 2.5$ [21], in perfect agreement with the numerical data [22], but no field theory expansion around this dimension exists. Moreover, practically there is no real space renormalization group approach that would be badly needed.

Why has progress has been so slow? The problem we face is very far from being simple; in the course of the investigations we discovered new phenomena that nobody was forecasting at the beginning of the investigations. It turned out that the order parameter in the mean-field approximation is a probability distribution on an infinite-dimensional space, which sounds more or less like a function of infinitely many variables [13, 23]: the free energy can be written as a function of this probability distribution. In the actual solution of the SK model one can restrict oneself to consider only a very small subspace of these probability distributions, i.e. the stochastically stable ultrametric distribution¹. Indeed the study of the SK model was the starting point of analytic computations and theoretical understanding of complex systems in many different areas of science [13, 24].

2.2. Some properties of mean-field theory

In mean-field theory there are many equilibrium states (an infinite number of states in the infinite volume limit). These states are more or less equivalent and they can be distinguished one from the another by considering their mutual overlap

$$q_{\alpha,\gamma} = N^{-1} \sum_{i=1,N} m(i)_\alpha m(i)_\gamma, \quad m(i)_\alpha \equiv \langle \sigma(i) \rangle_\alpha, \quad (1)$$

where $\langle \sigma_i \rangle_\alpha$ denotes the expectation value restricted to the state α .

¹ A definition of stochastic stability and ultrametricity will be presented later.

The properties of these states (e.g. the set of the values of their overlaps) change with the instance of the system. Analytically we can only compute their probability distribution, which is the order parameter of the problem as we have already seen.

For each system we define a function $P_J(q)$, with q being the overlap. Its average is a function $P(q)$ defined as

$$P(q) = \overline{P_J(q)}, \quad (2)$$

where the overline denotes the average of the couplings J .

The function $P(q)$ is also non-trivial in finite dimensions. Also the function $P_J(q)$ is non-trivial and it changes from system to system. All functions $P_J(q)$ have a delta peak at $q = q_{EA} \equiv q_{\alpha,\alpha}$ (the peak being obviously rounded for a finite volume system). The value of q_{EA} is independent of the state and this is due to their macroscopic indistinguishability.

Many phases are present for the same value of the parameters, therefore each point is a critical point (more precisely a multicritical point); sometimes (as it happens in the SK model) each state is at a second-order phase transition point and therefore long-range correlations are expected to be present inside each state [25, 26]. We have found here a completely unexpected realization of SOCE (self organized criticality at equilibrium).

The educated reader will object that this picture is not possible for generic Hamiltonians. This violates the Gibbs rule [27] that states that multicritical points (where $K + 1$ phases are present) exist on a manifold of codimension K in the parameter space. Apparently this picture seems to be not consistent: it would be unstable under generic perturbations unless something special happens. However Guerra [14] was able to turn the argument around in a brilliant way: *something special does happen*. The whole picture is possible only if some conditions are satisfied, i.e. the so-called stochastic stability identities found by Guerra. They can be derived by assuming that the system is stable against a generic perturbation.

2.3. The two susceptibilities

This picture has immediate experimental consequences. We can define two physically relevant susceptibilities.

- The linear response susceptibility χ_{LR} , i.e. the response within a state, that is observable when we change the magnetic field at fixed temperature and we do not wait for too much time². We have that

$$\chi_{LR} = \beta(1 - q_{EA}). \quad (3)$$

- The true equilibrium susceptibility χ_{eq} , that is very near to χ_{FC} , the field cooled susceptibility, where one cools the system in the presence of a field:

$$\chi_{eq} = \beta \left(1 - \int |q| P(q) dq \right). \quad (4)$$

The difference of the two susceptibilities is the hallmark of replica symmetry breaking. The experimental data are shown in figure 1.

3. Present theoretical understanding

3.1. General considerations

As we have already seen the actual solution of the SK model is a particular one, and in spite of its rather large complication is the simplest one for a system with many equilibrium states.

² More precisely a time that is smaller than a function $f(h)$ that diverges when the magnetic field h goes to zero.

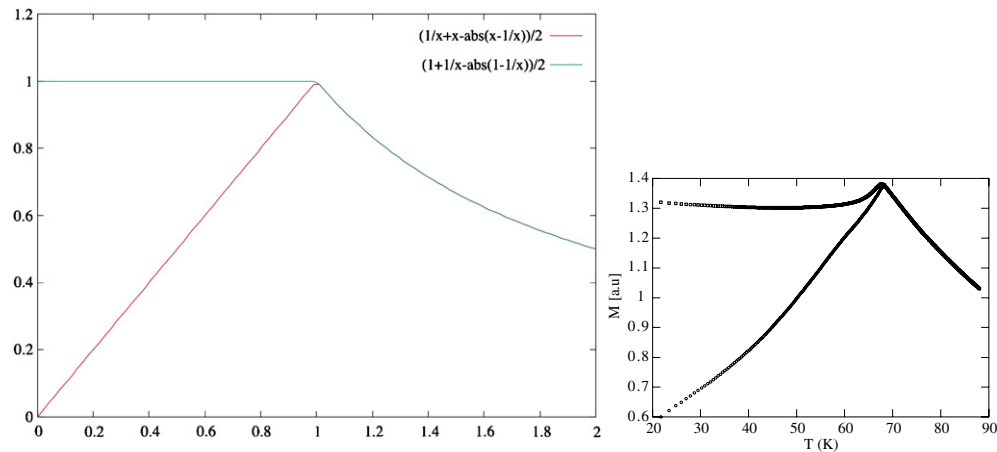


Figure 1. At the left we obtain the results for the SK model, at the right we have experimental data on metallic spin glasses [28]. The similarities among the two figures are striking.

In principle one can consider more complex mean-field solutions; sincerely I hope that these more complex solutions of the mean-field equations are not necessary for spin glasses, but they could be useful in other contexts (e.g. in non-equilibrium systems or in evolutionary systems with sexual reproduction [13]).

The basic principles, on which the theory is based, are the following:

- The function $P_J(q)$ is non-trivial and it changes from system to system.
- The system is stochastically stable in the presence of an infinitesimal magnetic field that breaks the spin reversal symmetry³.
- Overlap equivalence states that all other possible definitions of overlaps are equivalent to the original one, i.e. in the infinite volume limit the new overlaps become given functions of the usual overlap q [16].
- Ultrametricity, to be defined later.

These four principles are listed in order of relevance and generality. In principle it is possible that the last two could not be valid in sufficiently small dimensions and this would call for modifications of the field theory and (maybe) for a more complex order parameter.

It is clear that at sufficiently low dimensions (e.g. for dimensions less than 2.5) the order parameter must be zero and mean field does not apply anymore; however some remnants of mean-field theory may survive if the system is not too large or the observation time is not too long, as it happens for two-dimensional superconductors where the Landau Ginsburg theory applies very well at low temperatures on human scales, in spite of the absence of an order parameter at equilibrium.

3.2. The probability distribution of the overlap

There are no doubts from simulations, also in three dimensions (at least at zero magnetic field) that the function $P_J(q)$ is non-trivial and it fluctuates from system to system. In figure 2, we

³ The spin reversal symmetry produces a two-fold degeneracy of the equilibrium states that disappears in an infinitesimal magnetic field. It would also possible to eliminate the problem by considering $|q|$ at the place of q . See also the discussion in [45] on the sign of the product of the overlaps of three different replicas.

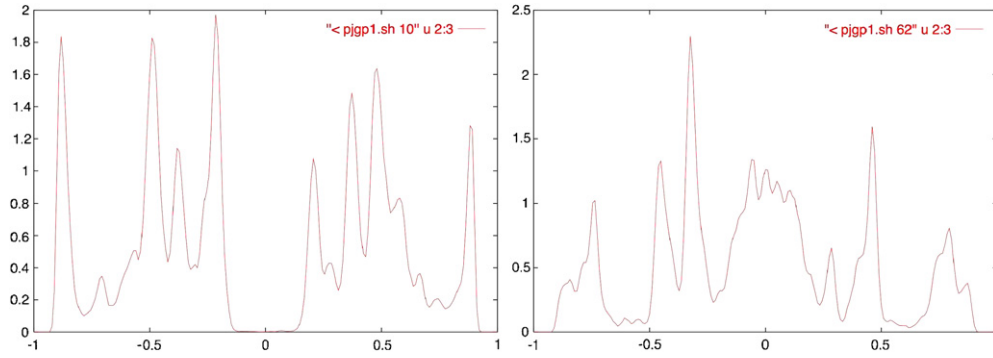


Figure 2. The function $P_J(q)$ for two samples (i.e. a different choice of J) for $D = 3, L = 16$ (16^3 spins) [29].

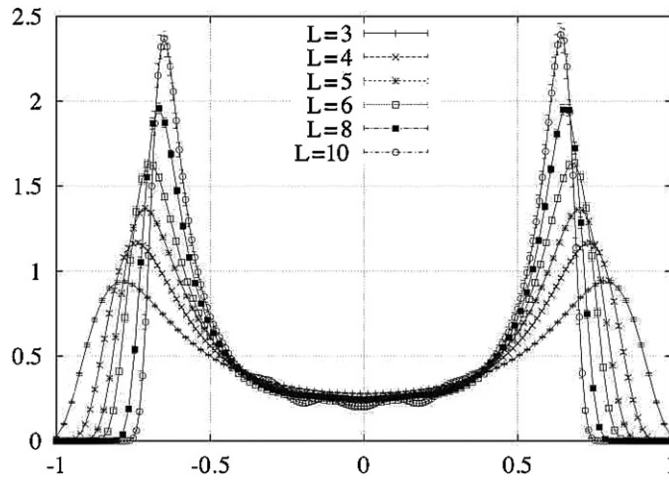


Figure 3. The function $P(q) = \overline{P_J(q)}$ for $D = 4$ after average over many samples ($L = 3, \dots, 10$) [31].

show some data for two three-dimensional samples with side $L = 16$. In figure 3, we show the average function $P(q)$ in four dimensions.

This effect has been consistently seen in all the simulations for a quite large range of temperatures (practically down to $T = 0$) and it also persists for larger lattices (the largest lattices, where systematic investigations have been done, have side $L = 24$ in dimensions 3). There are very strong indications that this effect should persist in the infinite volume limit. Also the critics of the replica approach accept this point, which is not controversial anymore.

3.3. Stochastic stability

Stochastic stability is a very strong principle: it implies the existence of an infinite set of identities, maybe the most well-known being

$$P(q_{1,2}, q_{3,4}) \equiv \overline{P_J(q_{1,2})P_J(q_{3,4})} = \frac{2}{3}P(q_{1,2})P(q_{3,4}) + \frac{1}{3}P(q_{1,2})\delta(q_{1,2} - q_{3,4}). \quad (5)$$

The function $P(q_{1,2}, q_{3,4})$ can be measured in simulations in a rather simple way by considering four identical replicas (or clones) of the same system (i.e. all with the same Hamiltonian). A direct test of this last relation is not simple, because delta function are rounded in finite volume system. Simpler relations are obtained by taking the moments. For example, let us consider four replicas of the system. The previous equation implies that

$$\overline{(q_{1,2}^2 q_{3,4}^2)} = \frac{2}{3} \overline{(q_{1,2}^2)^2} + \frac{1}{3} \overline{(q_{1,2}^4)}, \quad (6)$$

which is satisfied numerically with very high accuracy [32].

In some sense stochastic stability is not an assumption. If we consider the link overlap (to be defined later) at the place of the usual overlap, stochastic stability is a theorem in equilibrium system that has been recently proved [33]. Of course the theorem is valid in the infinite volume limit and finite volume corrections have not been evaluated, so that it is a welcome information to know that, at least for the low moments concerned in equation (6), the stochastic stabilities identities are satisfied with high precision also for not too large systems.

Stochastic stability is a very strong requirement, whose implication have not been completely spelled out. We only note here the very interesting fact that two non-interacting stochastically stable systems do not form a single stochastically stable system, so that stochastic stability implies in some sense the existence of a certain degree of coherence of the whole system. Moreover, let me mention (although this paper is devoted to statics) that stochastic stability gives the link to relate the equilibrium properties to the properties of systems that are slightly of equilibrium, recovering in this way the famous fluctuation dissipation relations [34–38]. More detailed consequences of stochastic stability are discussed in [39, 40].

3.4. Overlap equivalence

Of course there could be many definitions of overlaps [13]. You may take any quantity $O(i)$ and define an O -dependent overlap as

$$q_{\alpha,\gamma}^O = N^{-1} \sum_i \langle O(i) \rangle_\alpha \langle O(i) \rangle_\gamma. \quad (7)$$

A special case, that has been often investigated, is the link overlap that on a D -dimensional lattice is defined as

$$q_{\alpha,\gamma}^L = 1/(DN) \sum_{i,k} \langle \sigma(i) \sigma(k) \rangle_\alpha \langle \sigma(i) \sigma(k) \rangle_\gamma, \quad (8)$$

where the sum goes on all the DN links i, k of the cubic lattice (i.e. all the pairs of nearest neighbor points).

Overlap equivalence states that in the infinite volume limit

$$q_{\alpha,\gamma}^O = f^O(q_{\alpha,\gamma}), \quad (9)$$

where the function f^O may depend on the temperature. In the case of the SK model we have the simple relation

$$q_{\alpha,\gamma}^L = q_{\alpha,\gamma}^2. \quad (10)$$

This statement may be cast in a more explicit way if we consider the overlap correlation $G(x)$ defined in a system composed of two replicas of the same system (which for definiteness we call σ and τ):

$$G(x) = \langle \sigma(x) \tau(x) \sigma(0) \tau(0) \rangle. \quad (11)$$

We also define the constrained overlap correlation $G(x|q)$ where the previous average is done only on those pairs of configurations that have overlap q . It can be shown that overlap equivalence is equivalent to the statement that the correlation functions are clustering in this constrained ensemble [16] and therefore

$$\lim_{x \rightarrow \infty} G(x|q) = q^2. \tag{12}$$

It is interesting to note that the correlation at distance 1, $G(1|q)$, is also called the link overlap $Q_L(q)$. The previous relations implies that in the infinite volume limit

$$\langle q^O \rangle_q = f^O(q), \quad \langle (q^O)^2 \rangle_q = \langle q^O \rangle_q^2. \tag{13}$$

In the same way in the ferromagnetic Ising model the violations of clustering and the presence of fluctuations in intensive quantities are eliminated by considering the constrained ensembles with positive (or negative) magnetization, therefore the sign of the magnetization is the order parameter. Here the situation is similar, but the constraint is the value overlap. In the ensemble of two replicas at fixed overlap *intensive quantities do not fluctuate*. This principle partially closes the Pandora box opened by the fact that in the usual unconstrained ensemble intensive quantities do fluctuate [1].

This picture implies that the probability distribution of the window overlaps, i.e. overlaps in a region of side R when the side L goes to infinity first, is similar to that a system with size L , a crucial prediction that has been directly checked [32, 41].

This scenario has been questioned and an alternative scenario, the TNT scenario, has been proposed [7]. According to this scenario the situation should be similar to the Ising ferromagnetic with antiperiodic boundary conditions: there are many states (the interface may be anywhere) and there is a non-trivial $P(q)$. These states are locally identical (apart from a spin flip) with the exclusion of a region whose relative volume goes to zero as L^{-a} with $a = 1$. In the TNT scenario we would have that for $x \ll L$, $G(x|q)$ does not depend on q for not too large q , i.e. for $1 \ll x \ll L$

$$G(x|q) = q_{ea}^2, \tag{14}$$

while for x of $O(L)$, $G(x|q)$ is a function of x/L .

The TNT scenario, in spite of the difficulties to put it in a stochastically stable form⁴, had some popularity a few years ago, when it was proposed for the first time. It would predict a delta function probability distribution for the link overlap. However, these conclusions were based on not too large lattices. A more complete analysis already showed that the arguments for the TNT scenario were related to transient finite volume effects [42]. Indeed it is now very clear, both from data for small lattices but at low temperature [43] and from data on quite larger lattices (i.e. up to $L = 20$) [44, 45] that the TNT scenario is not correct and that for q not very near to q_{EA} we have

$$\langle Q^L(q) \rangle = A + Bq^2, \tag{15}$$

where B has a definite limit when L goes to infinity (see figure 4). Moreover the fluctuations of $Q^L(q)$, i.e. its variance, go to zero quite fast with the side (something like $L^{-1.4}$, see figure 5).

3.5. Dynamic correlations

A particularly interesting case is the correlations at $q = 0$. According to the clustering properties they should go to zero at infinity, as a power with an exponent that can be estimated analytically:

$$G(x|0) \propto x^{-\alpha}. \tag{16}$$

⁴ The TNT scenario is also in variance with the known properties of the window overlaps [32, 41].

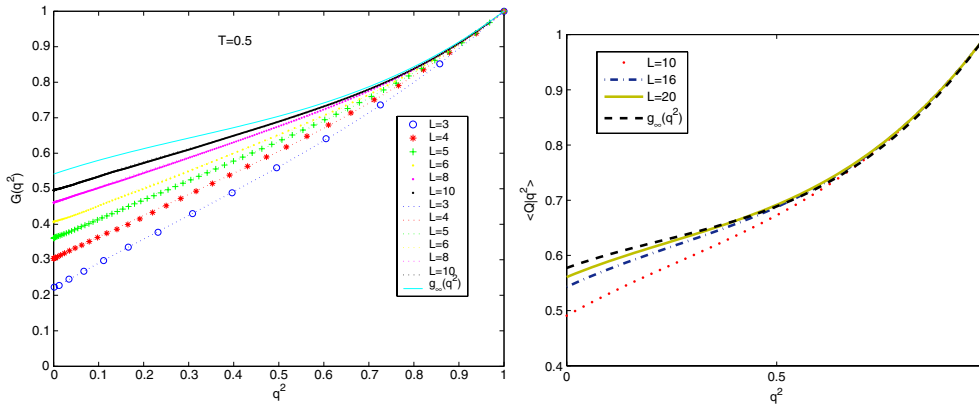


Figure 4. Plot of the curves $Q_L(q^2)$ (for $T = 0.5$ Gaussian model): left panel $L = 3, 10$ and infinite volume extrapolation [44]. Right panel: plot of the curves $Q_L(q^2)$ (for $T = 0.5$ bimodal model): $L = 10, 20$ and infinite volume extrapolation [45].

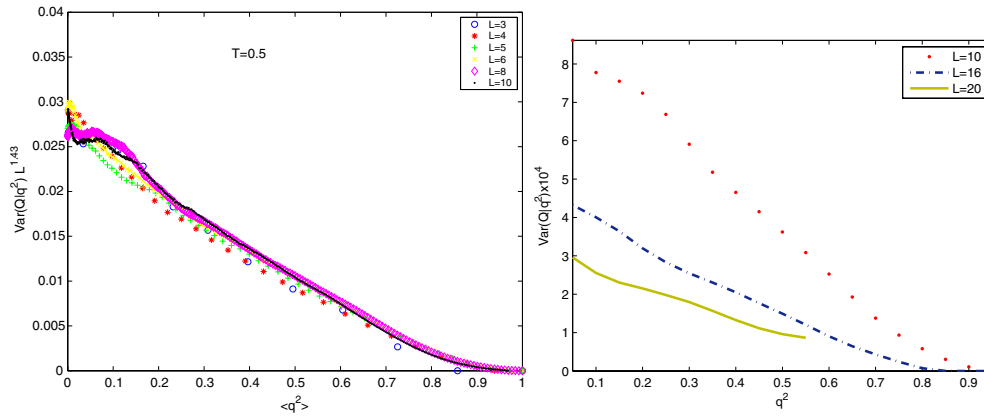


Figure 5. Variance of $Q_L(q^2)$ (left panel divided by $L^{1.43}$ for the temperature $T = 0.5$ [44], right panel not divided by $L^{1.43}$ for the temperature $T = 0.7$ [45]), as a function of q^2 , for different sizes L .

The interest in these correlations is due to the fact that they can be computed in the dynamics [46]: indeed if one consider a pair of two large systems, if $q = 0$ at time zero, q remains zero at all times. Therefore, one can wait for a long time and compute the equal time correlation function. Obviously, they will remain nearly equal to zero at distances larger than the dynamically growing correlation length $\xi(t)$. By definition in the limit $t \rightarrow \infty$ and fixed x we obtain equilibrium correlations in some equilibrium state.

In the TNT scenario, the correlation functions are the same in all equilibrium states in the infinite volume limit and for not too small x they are given by $G(x) = q_{EA}^2$; therefore we must have

$$G(x, t) = q_{EA}^2 f(x/\xi(t)). \tag{17}$$

In contrast, if overlap equivalence holds we should have

$$G(x, t) = x^{-\alpha} f(x/\xi(t)). \tag{18}$$

There is in the literature no indications whatsoever that equation (17) holds for the correlations;

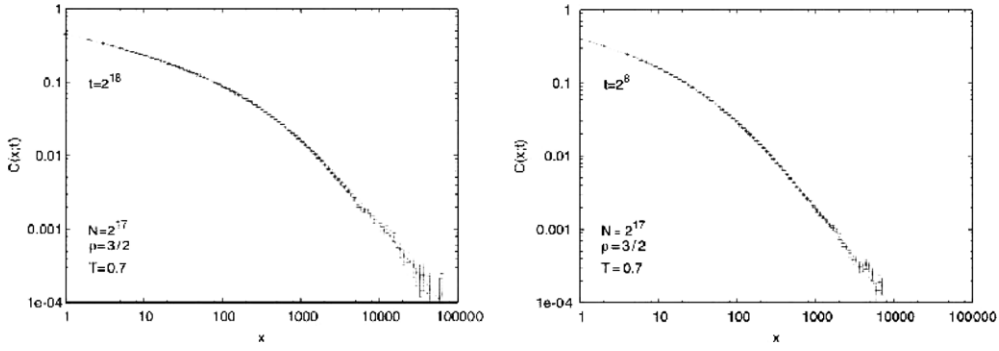


Figure 6. The correlation functions on a one-dimensional lattice with $N = 2^{17}$ and $\rho = 1.5$ in the low temperature phase ($T \approx 0.4T_c$) at time $t = 2^{18}$ (left panel) and at time $t = 2^8$ (right panel) as a function of the distance, fitted with formula (20).

all numerical simulations (which have been done in quite diverse situations) consistently support equation (18).

However someone may object that in most cases the correlation functions are measured not at too large distances, so that a fit may be ambiguous. A very interesting test of the theoretical predictions can be done in one-dimensional models with long-range interactions. If we use the diluted version [48] of Young’s long-range model [47], one can easily measure the correlation functions at distances 10^5 (linear lattices with $N = 10^6$ sites can be simulated). The model is such that

$$\overline{J(i, k)^2} \propto |i - k|^{-\rho}. \tag{19}$$

More precisely in the diluted version $J_{i,k} = \pm 1$ with probability $|i - k|^{-\rho}$, otherwise it is zero [48].

This model is a proxy of the D -dimensional short-range model, with the relation $\rho = 1 + 2/D$, in the sense that the two models have the same upper critical dimension, although they may have a different lower critical dimension. The data may be very well fitted for four decades in x by

$$G(x, t) = ax^{-\alpha} (1 + (x/\xi(t))^{\delta(\rho-\alpha)})^{-1/\delta}, \tag{20}$$

where we have imposed at very large distances $G(x, t) \propto x^{-\rho}$ (see, for example, figure 6). It turns out that α is roughly temperature independent below and not near to T_c . The correlation function (at least at some temperatures) seems to increase as a stretched power, i.e. as $\xi(t) \propto B \exp(A \log(t)^{1/2})$.

The conclusion is quite clear: the dynamic correlation function behaves in a way that is quite different from that of the droplet model and TNT scenario and they are perfectly consistent with the clustering properties of the replica approach. Two different copies evolve locally toward two different incongruent ground states.

4. Ultrametricity

Ultrametricity states a very striking property for a physical system: essentially it says that the equilibrium configurations of a large system can be classified in a taxonomic (hierarchical) way (as animals in different taxa): configurations are grouped in states, states are grouped in families, families are grouped in superfamilies.

Ultrametricity implies that sampling three configurations with respect to their common Boltzmann–Gibbs probability the distribution of the distances among them is supported, in the limit of very large systems, only on equilateral and isosceles triangles with no scalene triangles contribution.

We define the three replicas overlap distribution \mathcal{P}_3 :

$$\mathcal{P}_3(q_{1,2}, q_{2,3}, q_{3,1}) = \langle \delta(q_{1,2} - q(\sigma, \tau)) \delta(q_{2,3} - q(\tau, \gamma)) \delta(q_{3,1} - q(\gamma, \sigma)) \rangle, \quad (21)$$

where σ, τ, γ denote three different equilibrium configurations.

Ultrametricity implies that \mathcal{P}_3 is zero when the following inequality is violated:

$$q_{1,3} \geq \max(q_{1,2}, q_{2,3}). \quad (22)$$

Stochastic stability implies [49] an exact formula for $\overline{\mathcal{P}_3}$:

$$\begin{aligned} \overline{\mathcal{P}_3}(q_{1,2}, q_{2,3}, q_{3,1}) &= \frac{X(q_{1,2})}{2} P(q_{12}) X(q_{1,2}) \delta(q_{1,2} - q_{2,3}) \delta(q_{2,3} - q_{3,1}) \\ &\quad + \frac{1}{2} P(q_{1,2}) P(q_{2,3}) \theta(q_{1,2} - q_{2,3}) \delta(q_{2,3} - q_{3,1}) \\ &\quad + \frac{1}{2} P(q_{2,3}) P(q_{3,1}) \theta(q_{2,3} - q_{3,1}) \delta(q_{3,1} - q_{1,2}) \\ &\quad + \frac{1}{2} P(q_{3,1}) P(q_{1,2}) \theta(q_{3,1} - q_{1,2}) \delta(q_{1,2} - q_{2,3}), \end{aligned} \quad (23)$$

where $X(q) = \int_0^q P(q') dq'$.

There are some indications that suggest that for a stochastically stable system overlap equivalence should imply ultrametricity [50], however no mathematical theorem exists.

There were in the literature some direct tests of ultrametricity in dimensions 4 [51] and some indications (coming from the dynamics [52]) of the validity of ultrametricity in $D = 3$. It is therefore interesting to test directly if ultrametricity is present in $D = 3$.

There have been some simulations in which the ultrametricity property was investigated [53] by trying to identify the states and to define an ultrametricity index K that takes values in the interval $[0-1]$ such that $K = 0$ corresponds to ultrametricity. The simulations were done at low temperature in small lattices.

The expectation values of K as a function of the size of the system are shown in figure 7 both for the SK model and the three-dimensional Ising model. Both data are well fitted by a power of the volume

$$\langle K \rangle \propto N^{-\lambda}, \quad (24)$$

with $\lambda = 0.31$ in the SK model and $\lambda = 0.17$ in $d = 3$ (in other words $K \propto L^{-0.5}$). The data suggest that $\langle K \rangle$ goes to zero in the infinite volume limit also in $D = 3$ although for $L = 8$ the maximum lattice explored the value of K is not small. The reader should note that these are *not* the conclusions of the authors of the paper [51]: they concentrated their attention on the expectation of the value of K restricted to those configurations which have $K < 1$ and with this restriction the behavior is not so simple. Although the difference between the two quantities does not matter in the limit where K goes to zero, the restriction of considering only the data with $K < 1$ introduces a finite volume distortion and it may hide a simple power behavior.

Since the extrapolation $K = 0$ starting from large values of K could be dangerous, it is interesting to directly study the ultrametricity on a larger lattice, also using a less sophisticated analysis that has a simple direct interpretation.

This was done in [45]. We considered three independent configurations 1, 2, 3 such that

$$|q_{1,2}| < |q_{1,3}| < |q_{2,3}|. \quad (25)$$

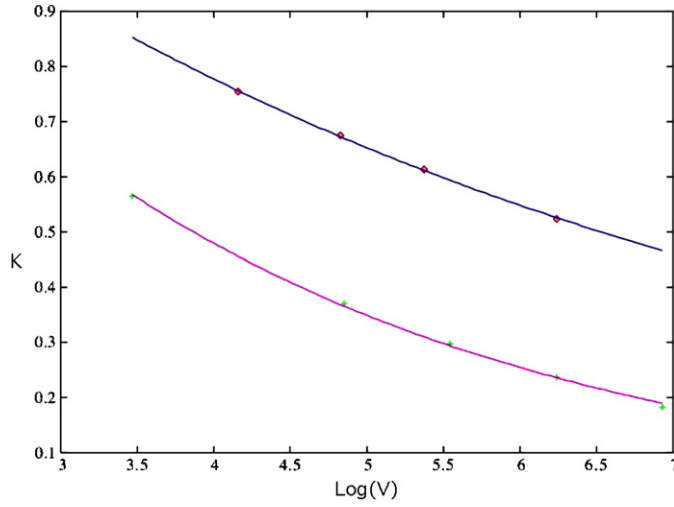


Figure 7. Data extracted from [51]: the average value of K (including data with $K = 1$, as a function of the volume in the SK model (lower curve) and in the 3D EA model (upper curve). The curves are power fits to the data.

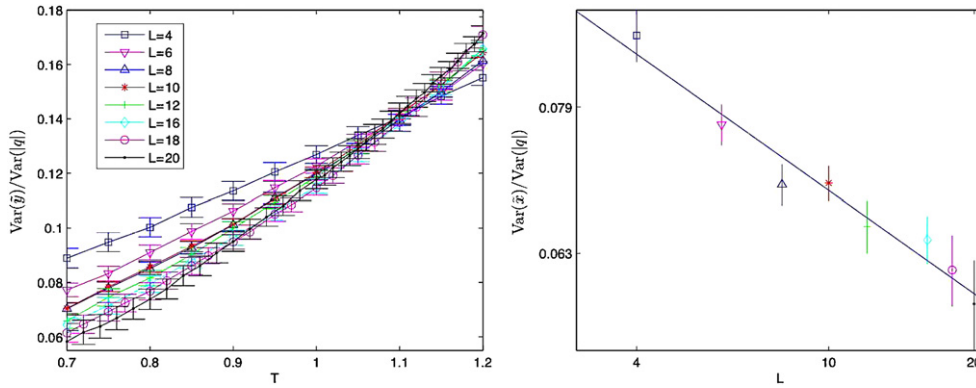


Figure 8. The ultrametricity index U as a function of the temperature and of the volume (left panel); the curves crosses at the critical point. The ultrametricity index U as a function of the size of the lattice at $T = 0.7$ (right panel), where the line is a fit proportional to $L^{-0.3}$.

If we flip the configurations in such a way that $0 < q_{1,3} < q_{2,3}$, it turns out that below the critical temperature $q_{1,2}$ is positive or, if negative, very near to zero. As far as ultrametricity implies that $q_{1,2} = q_{1,3}$ a natural ultrametricity index is given by

$$U = \frac{\langle (q_{1,2} - q_{1,3})^2 \rangle}{\langle q^2 \rangle}. \quad (26)$$

We have measured U for lattices going from $L = 4$ to $L = 20$ for temperatures down to $0.6T_c$ (see figure 8).

At T_c the probability distribution is not ultrametric but is nearly ultrametric: the value of U at the fixed point is much smaller than the value at the high temperature Gaussian fixed point (i.e. around 0.8) Ultrametricity seems to be present below T_c , but it sets in slowly like $L^{-0.3}$. It may be possible that the small power of L is an artifact of being too near to the critical

temperature and that the exponent of the power may be slightly high at smaller temperatures, showing a faster approach to ultrametricity.

5. Conclusions and questions to be answered

The replica symmetry breaking picture is confirmed. Good evidence is found for overlap equivalence and alternative interpretative schemes like the droplet model or the TNT scenario are no more viable.

Ultrametricity seems to be present in the statics. However, the measurements are difficult, there are strong finite volume corrections. In the future, we must recheck the $D = 4$ short-range model and study the long-range 1D model for different values of ρ . Given the slow approach to ultrametricity in the statics one may wonder what happens in the dynamics. Can one use the measured violation of ultrametricity at finite volume in the statics to compute the violations of dynamical ultrametricity in the dynamics?

A very important issue is the behavior of a spin glass in a magnetic field. It is possible that in the same way that a random magnetic field destroys the ferromagnetic order in $D = 2$, a random (or constant) magnetic field may destroy spin glass order in $D = 3$.

In the last years some simulation papers appeared claiming that the de Almeida line, that should separate the replica exact from the replica symmetry broken regions is not present in $D = 3$ [47, 54]. I would be more prudent to reach these conclusions:

- For finite volume there is a cross-over region where configurations with negative magnetization have also an important role in the presence of a small non-zero magnetic field. Any analysis done inside this region may be quite dangerous.
- There are analyses done on a very large lattice with a different methodology that indicate that a transition is present in a magnetic field and these analyses cannot be easily dismissed [55].
- The previous analysis [55] showed that in the presence of a magnetic field the effects of replica symmetry breaking are strongly decreasing with decreasing the dimension, so it is not clear if the analysis that miss the dAT transition has enough sensitivity to detect it.

This problem should be carefully investigated and now we have all the tools to investigate the problem more deeply. It remains however a problem quite difficult to be studied, the window of magnetic fields (not too small, not too big) for which sensible results may be obtained may be small in simulations.

Generally speaking analytic progresses are also deeply needed: the development of a nonlinear sigma model at low temperature and of a real space renormalization group results are very important steps that should be achieved.

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